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SIMULATION OF STEADY COMPRESSIBLE FLOWS BASED ON CAUCHY/RIEMANN EQUATIONS AND CROCCO'S RELATIONS PART II. VISCOUS FLOWS

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SUMMARY

In this paper, an alternative formation of the Navier–Stokes equations is used for steady compressible fluid flows. Two scalar equations are solved for the total enthalpy and entropy respectively, and a system of equations is solved for the velocity components. The latter consists of the continuity equation and the definitions of vorticity. Numerical results for two-dimensional low and high Reynolds number flows are presented and compared with standard calculations. The advantages of the present formulation are briefly discussed. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: compressible viscous flows; airfoils; Cauchy/Riemann equations; Crocco's relations

1. INTRODUCTION

In a previous paper [1], the authors used a formulation based on the Cauchy/Riemann equations and Crocco's relations to simulate steady compressible inviscid flows and the results were in agreement with the solutions obtained from standard Euler codes. The motivations to use this formulation are the following. In the far-field, the total enthalpy and the entropy are uniformly constant in most aerodynamic applications, hence the calculations are reduced to the equations of the velocity components and the pressure is obtained from Bernoulli's law under the isentropic conditions. Moreover, the velocity equations can be replaced by a single potential equation in the far-field (except in the wake). If staggered grids are used, a discrete potential exists and the results of the potential calculations are identical to those obtained from the velocity equations. At any rate, the velocity equations can be easily imbedded in an artificial time-dependent symmetric hyperbolic system. Efficient and accurate methods can be used to solve such a system. A genuinely multi-dimensional scheme can be used for the scalar equations of the entropy and total enthalpy in the near-field using either structured or unstructured grids. The formulation allows for adaptive modeling of the physics and domain decomposition techniques can be effectively implemented.

In the present work, the extension to viscous flow simulation is considered. Now, one can think of three layers: the viscous layer (i.e. boundary layers and wakes), an inviscid rotational

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flow layer and the far-field. The length scales of the viscous and inviscid layers are different, particularly for high Reynolds number flows, hence the grid sizes in these layers are different and are chosen to resolve the physical phenomena in each layer (i.e. shock waves, separation, etc.). Moreover, the viscous and heat transfer terms are discretized only on the fine mesh of the inner layer. In the inviscid layer such terms are dropped. The magnitude of shear stress can be used as a criteria to switch from the viscous layer to the inviscid rotational flow layer, while the magnitude of vorticity or entropy variation can be used, to switch to the far-field layer. No special treatment is required for the interfaces between layers; only the non-homogeneous terms will be dropped based on the above criteria.

In the following, the governing equations of the present formulation are derived, and numerical methods and numerical results are discussed.

2. GOVERNING EQUATIONS

The kinematics of the flow can be determined from the continuity equation and the definition of vorticity, namely

$$(\rho u)_x + (\rho v)_y = 0, \tag{1}$$

$$-u_y + v_x = \omega. \tag{2}$$

The same equations are used for viscous as well as inviscid flows. To solve these equations for the velocity components u and v, ρ and ω should be given. The density is calculated in terms of the speed, the entropy and the total enthalpy, while the vorticity is calculated from Crocco's relation (including the viscous terms). The latter can be derived from the normal momentum equation. It can be shown that the density and the pressure can be written in the from

$$\rho = \rho_i \, \mathrm{e}^{-\Delta S/R}, \tag{3}$$

$$p = p_i \, \mathrm{e}^{-\Delta S/R}, \tag{4}$$

where

$$p_i = \frac{\rho_i^{\gamma}}{\gamma M_{\infty}^2}. \tag{5}$$

Inserting (3) and (4) in the definition of the total enthalpy, one obtains

$$\frac{1}{\gamma - 1} \frac{\rho_i^{\gamma}}{M_{\infty}^2} + \frac{1}{2}(u^2 + v^2) = H, \tag{6}$$

or

$$\rho_i = \left[(\gamma - 1) M_{\infty}^2 \left(H - \frac{1}{2}(u^2 + v^2) \right) \right]^{1/(\gamma - 1)}. \tag{7}$$

The entropy and the total enthalpy can be obtained from the tangential momentum equation and the energy equation respectively. The momentum equation can be written in the form

$$\rho \vec{q}_t + \rho \vec{q} \cdot \nabla \vec{q} = -\nabla p + \frac{1}{Re} \nabla \cdot \tau, \tag{8}$$

where τ is the viscous stress tensor.

Using the vector identity

$$\vec{q}\cdot\nabla\vec{q}=\nabla\left(\frac{1}{2}q^{2}\right)+\vec{\omega}\times\vec{q} \tag{9}$$

and the thermodynamic relation

$$T\nabla S=\nabla h-\frac{1}{\rho}\nabla P, \tag{10}$$

where S is the entropy and h is the specific enthalpy, one obtains

$$\vec{q}_{t}+\vec{\omega}\times\vec{q}=T\nabla S-\nabla H+\frac{1}{\rho Re}\nabla\cdot\tau, \tag{11}$$

where $H=h+1/2q^{2}$.

If the viscosity coefficient is assumed constant, independent of the temperature, the viscous term becomes

$$\nabla\cdot\tau=\left[\nabla^{2}\vec{q}+\frac{1}{3}\nabla(\nabla\cdot\vec{q})\right]. \tag{12}$$

The last term of Equation (12) vanishes only for incompressible flows. For steady two-dimensional flows, using Crocco's natural co-ordinates s and n (where s is the local stream line direction and n is normal to it), Equation (11) leads to Crocco's relation

$$\omega=\left(T\frac{\partial S}{\partial n}-\frac{\partial H}{\partial n}+\text{viscous terms}\right)\Big/q, \tag{13}$$

where ω is given by

$$\omega=\frac{q}{R}-q_{n}, \tag{14}$$

and R is the streamline radius of curvature. It is clear from Equation (13) that the vorticity can be generated due to entropy or total enthalpy normal gradients as well as from viscous stresses. In the shock region, one should use the momentum equations in conservation form; the normal momentum equation is given by

$$\vec{n}\cdot\left[\nabla\rho\vec{q}\vec{q}+\nabla p-\frac{1}{Re}\nabla\cdot\tau\right]=0, \tag{15}$$

where

$$\vec{n}=\left(-\frac{v}{q},\frac{u}{q}\right).$$

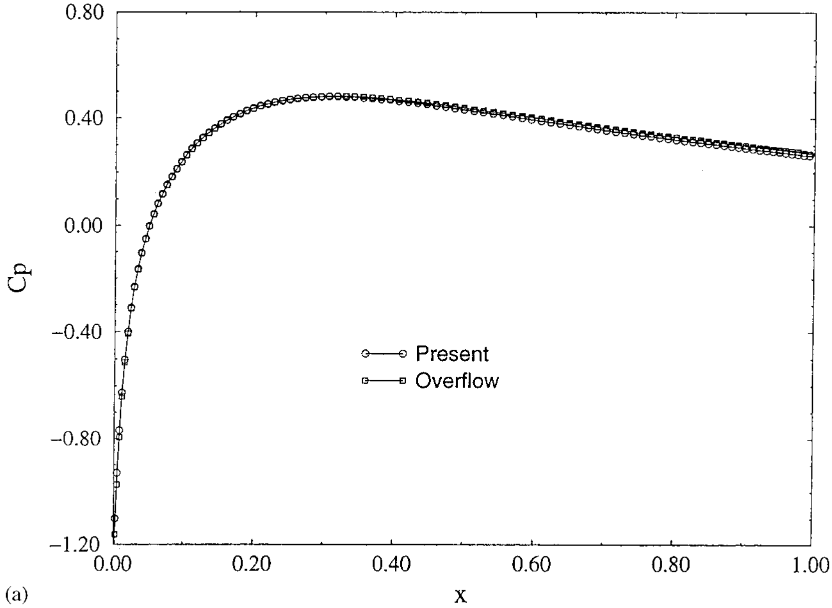
Similarly, the tangential momentum equation is

$$\vec{s}\cdot\left[\nabla\rho\vec{q}\vec{q}+\nabla p-\frac{1}{Re}\nabla\cdot\tau\right]=0, \tag{16}$$

where

$$\vec{s}=\left(\frac{u}{q},\frac{v}{q}\right).$$

It is clear from the non-conservative form



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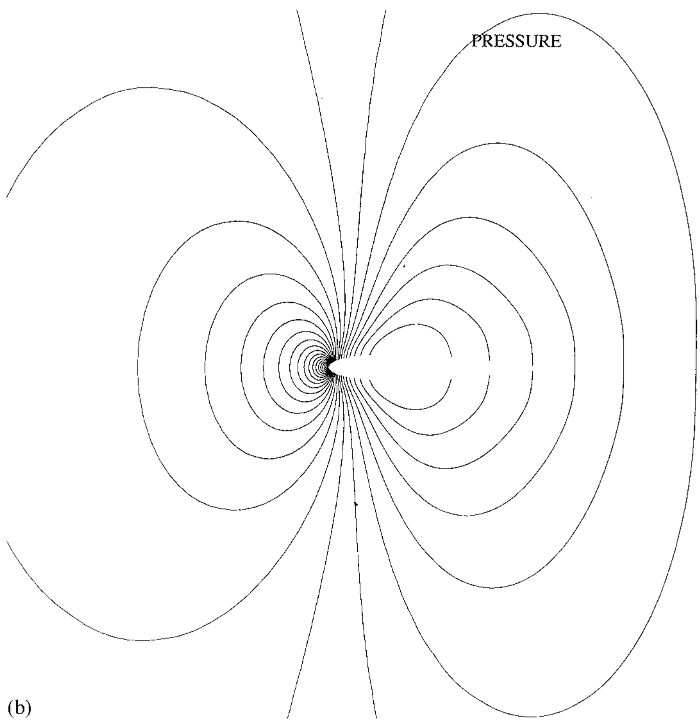
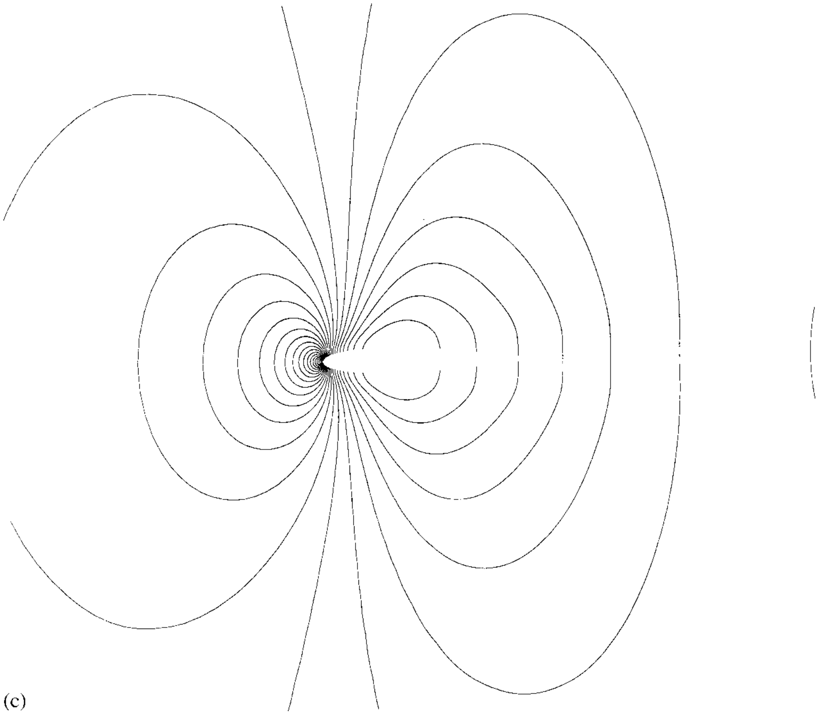


(b)

Figure 1. Results for NACA 0012, $\alpha = 0$, $M_\infty = 0.8$ and $Re = 500$. (a) Surface pressure distributions; (b) Pressure contours (Overflow); (c) pressure contours (present formulation); (d) Mach contours (Overflow); (e) Mach contours (present formulation).



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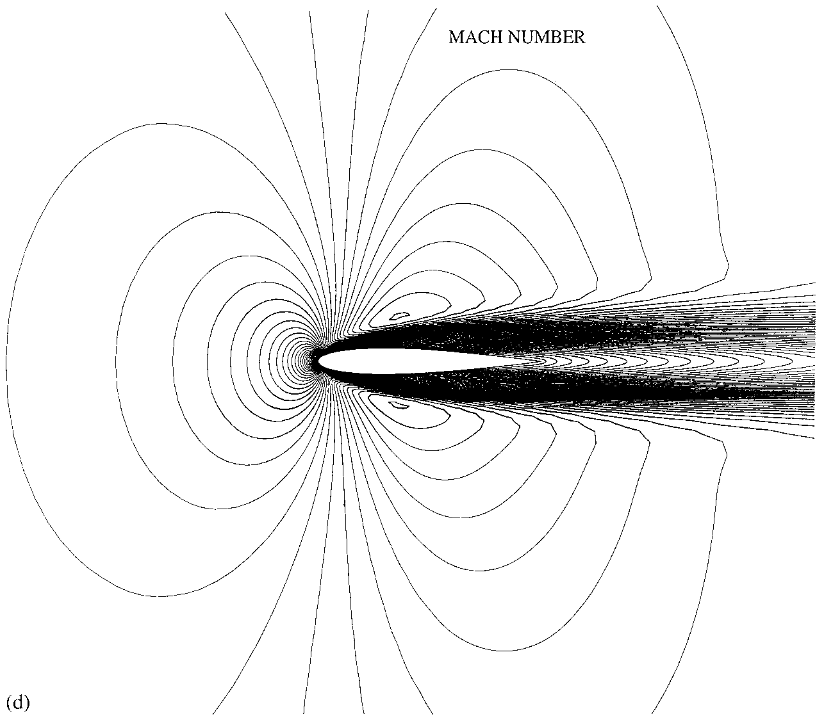

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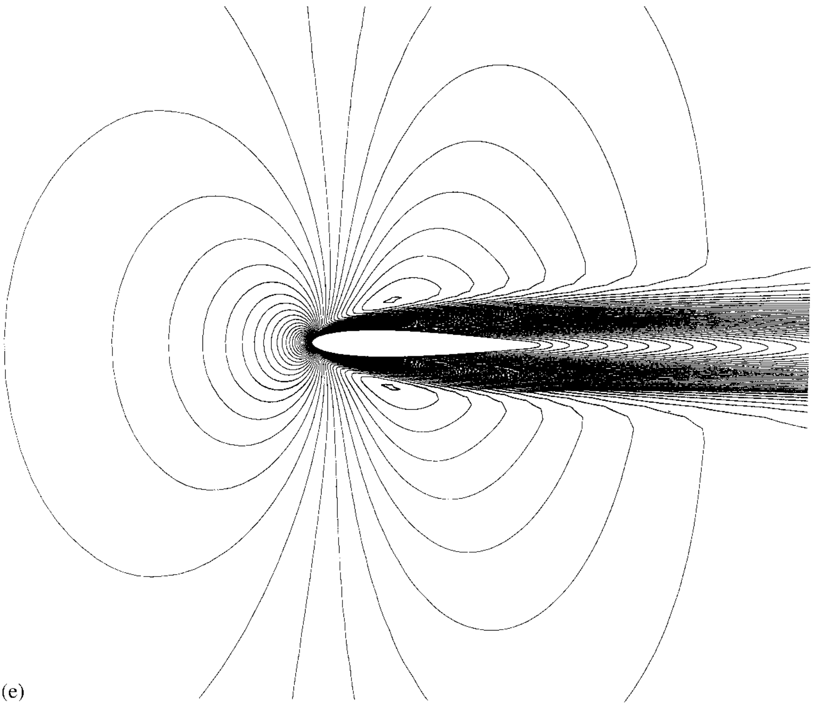

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Figure 1 (*Continued*)

$$\vec{s}\cdot\left[\vec{\omega}\times\vec{q}-T\nabla S+\nabla H-\frac{1}{\rho Re}\nabla\cdot\tau\right]=0, \qquad (17)$$

that entropy will remain constant along a streamline only for smooth inviscid flow. (Notice the first term vanishes identically since $\vec{s}=\vec{q}/|\vec{q}|$.)

Equation (17) (or Equation (16) in the shock region) can be solved for S or $E=\mathrm{e}^{-\Delta S/R}$, ($\Delta S=S-S_\infty$). The energy equation for steady flows can be written as (see [2])

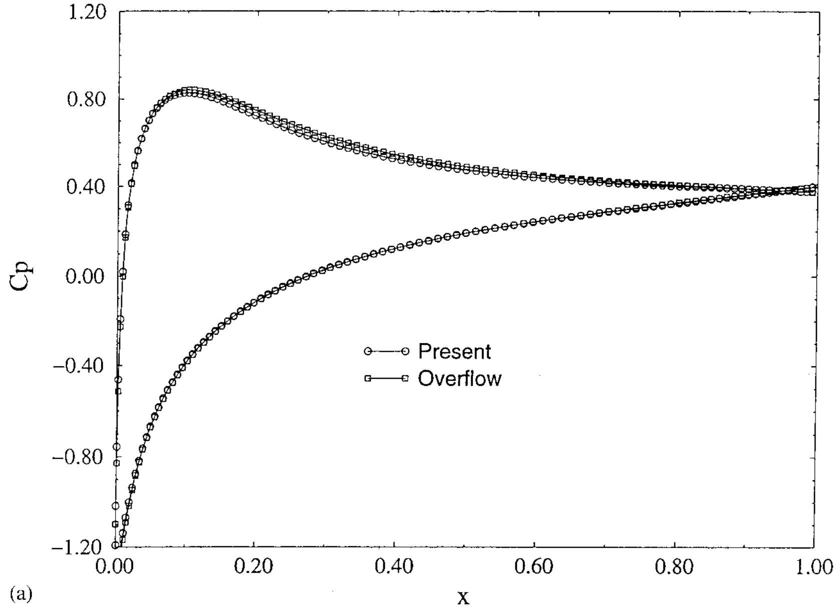
$$\nabla\cdot\rho\vec{q}H=\frac{1}{Pr\cdot Re}\nabla\cdot\bar{k}\nabla T+\frac{1}{Re}\nabla\cdot\vec{q}\tau. \qquad (18)$$

Equation (18) can be solved for the temperature assuming $h=c_pT$ or for H. If heat transfer and viscous terms are neglected, H remains constant along a streamline. This is true even across shock waves since $H=$ constant satisfies the Rankine–Hugoniot relations. Furthermore, if all streamlines originate from a uniform upstream reservoir and closed streamlines are not allowed, H is constant everywhere. It can be shown that for high Reynolds number flows with Prandtl number $Pr=1$, $H=H_\infty$ is an admissible solution of the boundary layer equations. In this case,

$$H=CpT+\frac{1}{2}u^2=CpT_\infty+\frac{1}{2}q_\infty^2=CpT_0. \qquad (19)$$

(We neglected $1/2u^2$ compared with $1/2v^2$ in the boundary layer.)

If Equation (19) is differentiated with respect to n, where n is in the direction normal to the wall, one obtains



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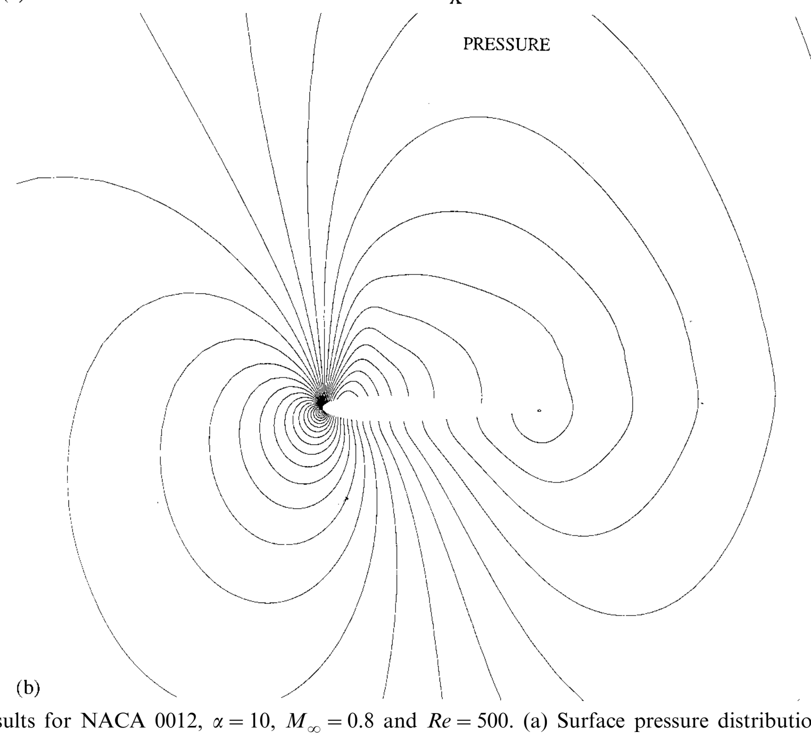
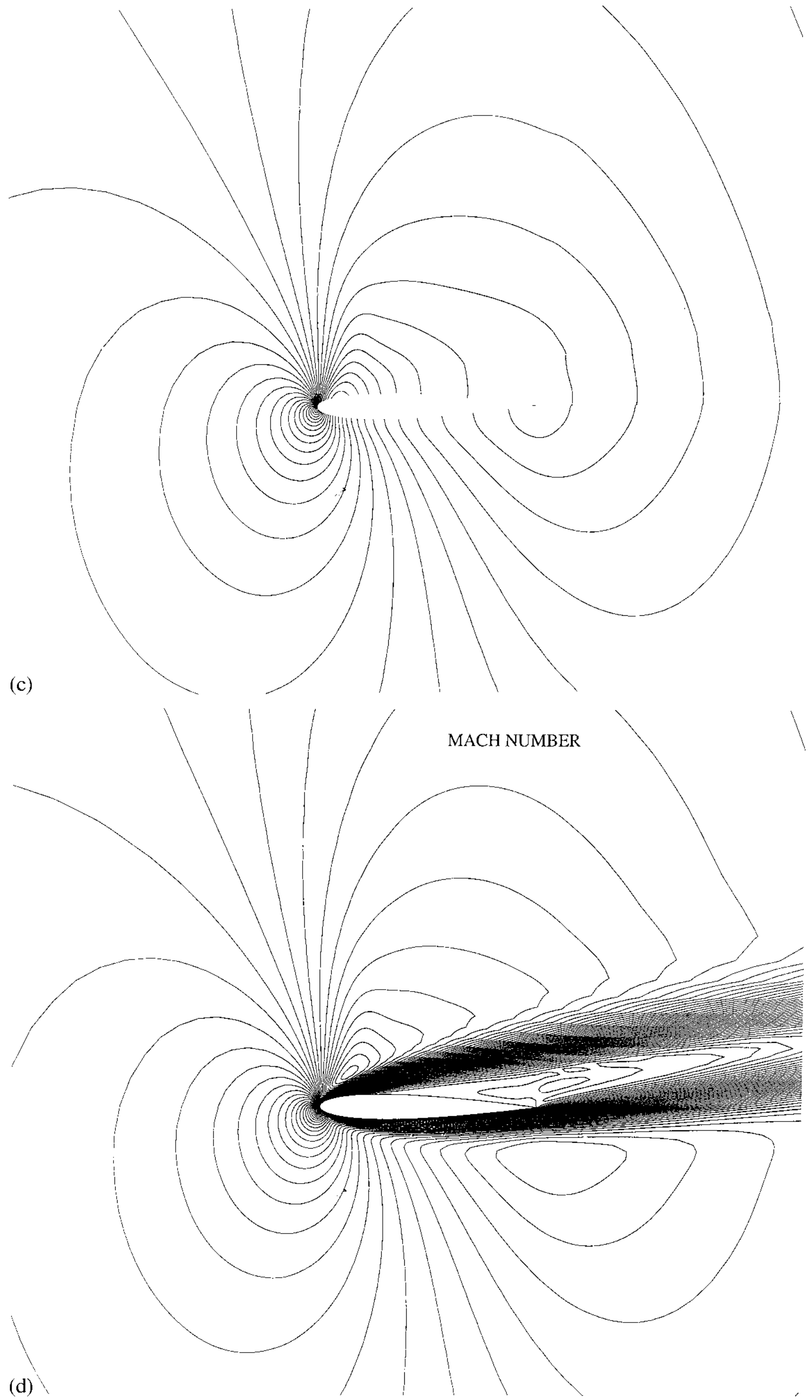


(b)

Figure 2. Results for NACA 0012, $\alpha = 10$, $M_\infty = 0.8$ and $Re = 500$. (a) Surface pressure distributions; (b) pressure contours (Overflow); (c) pressure contours (present formulation); (d) Mach contours (Overflow); (e) Mach contours (present formulation).



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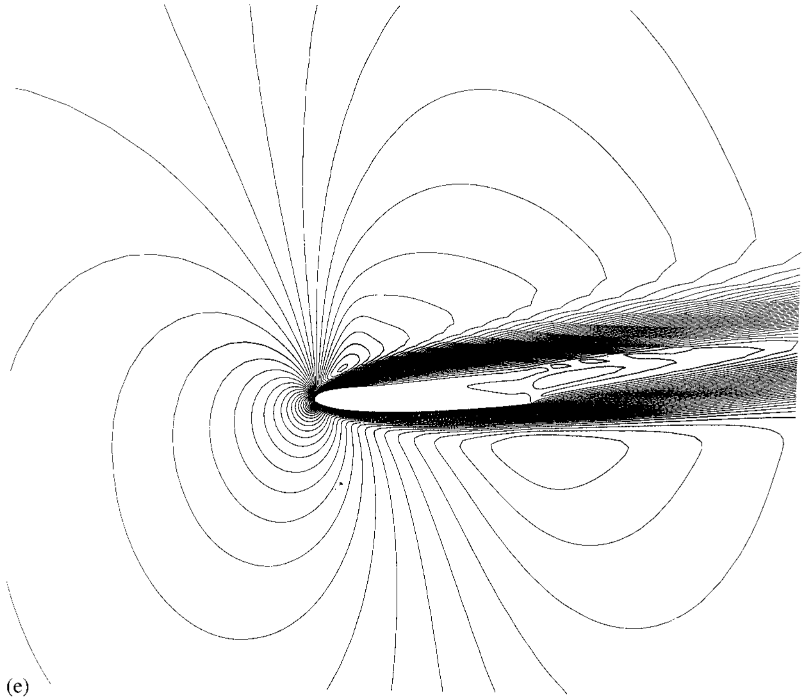


Figure 2 (*Continued*)

$$u\frac{\partial u}{\partial n} + Cp\frac{\partial T}{\partial n} = 0. \tag{20}$$

Equation (20) implies that heat transfer vanishes at the wall, i.e. the wall is insulated, and the wall temperature is the stagnation temperature T_0.

Therefore, in this case $H =$ constant is an asymptotic solution of the Navier–Stokes equations for steady compressible flows in the limit of high Reynolds numbers. For finite Reynolds numbers, however, $H =$ constant does not satisfy the energy equation but it is a good approximation. (It is not a valid approximation, however, for incompressible flow (constant density), where $H = p/\rho + 1/2u^2$. Obviously if the pressure does not vary across the boundary layer, H cannot be constant.)

In the present work, the energy equation is replaced by the assumption that $H =$ constant. The results are in agreement with the standard calculations based on the full energy equation. This simplification is independent of the present formulation and the energy equation can always be solved as a scalar equation for the temperature, even for the case of $Pr = 1$ with zero heat transfer or insulated surface.

The proposed approximation, however, simplifies the calculation of both the entropy and the vorticity since the gradient of the total enthalpy vanishes identically. Acceptable results are obtained even for low Reynolds number ($Re \sim 500$).

In summary, the difference between the present work and the previous effort on the inviscid flow are the viscous terms appearing in the normal and the tangential momentum equations. The latter affects the calculation of the entropy while the former affects the magnitude of vorticity in the viscous layer.

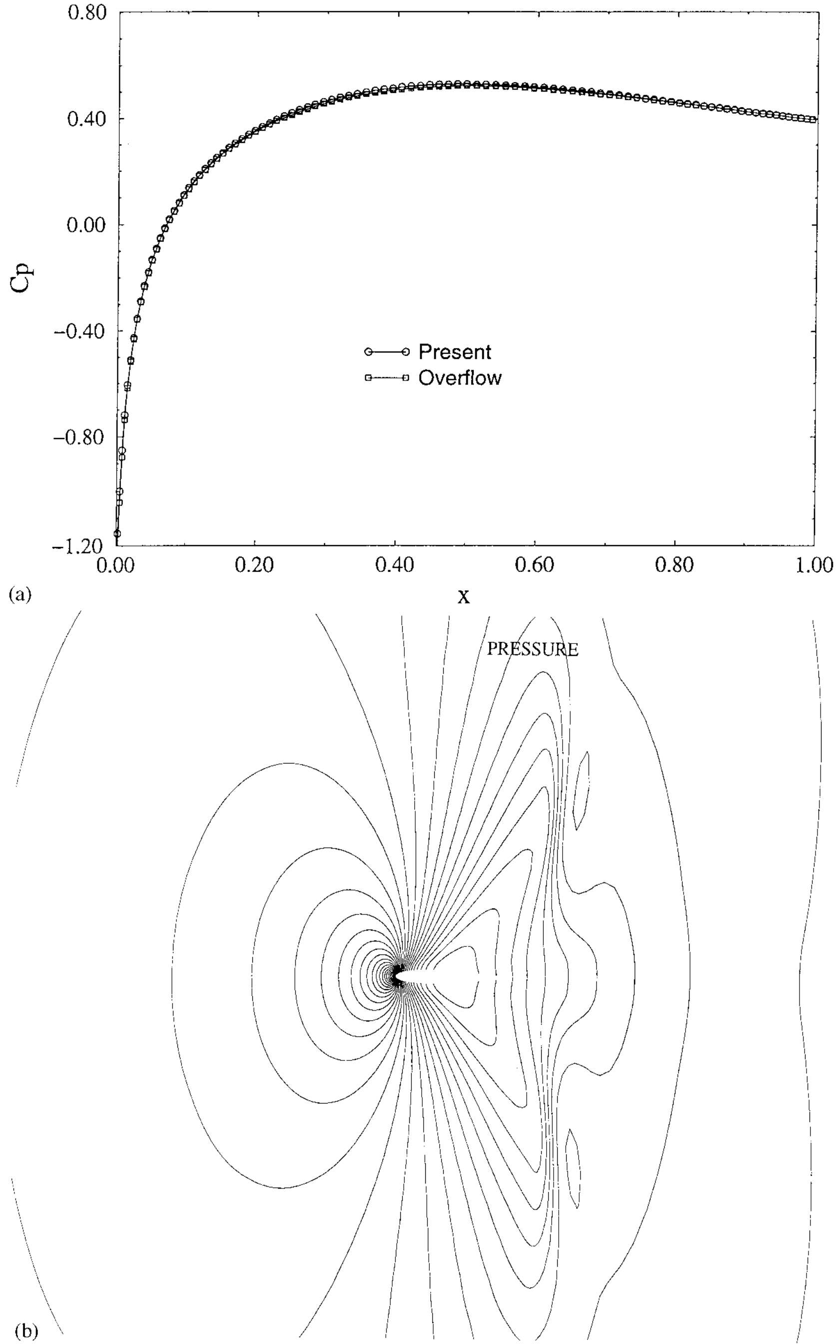
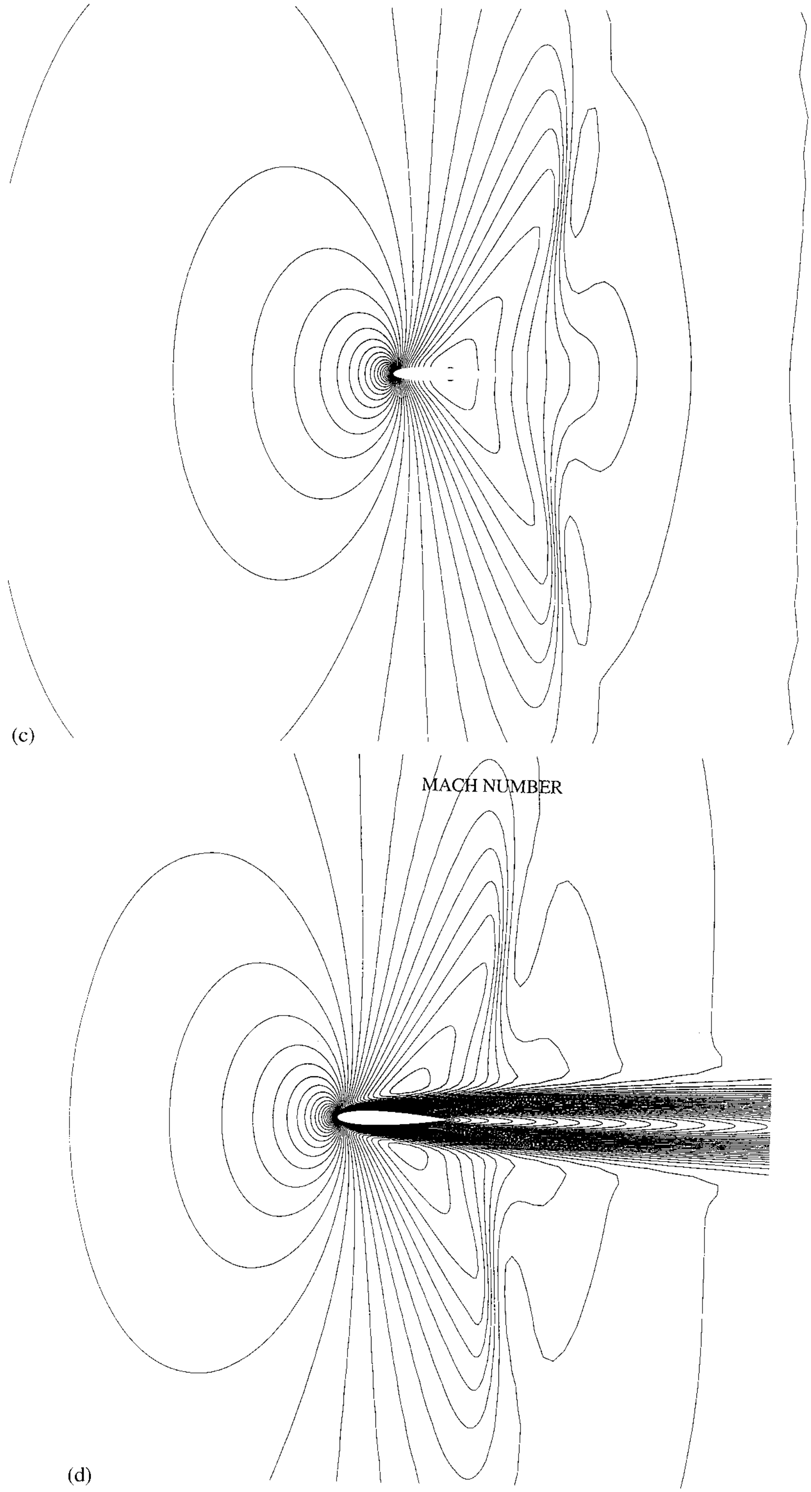


Figure 3. Results for NACA 0012, $\alpha = 0$, $M_\infty = 0.9$ and $Re = 500$. (a) Surface pressure distributions; (b) pressure contours (Overflow); (c) pressure contours (present formulation); (d) Mach contours (Overflow); (e) Mach contours (present formulation).



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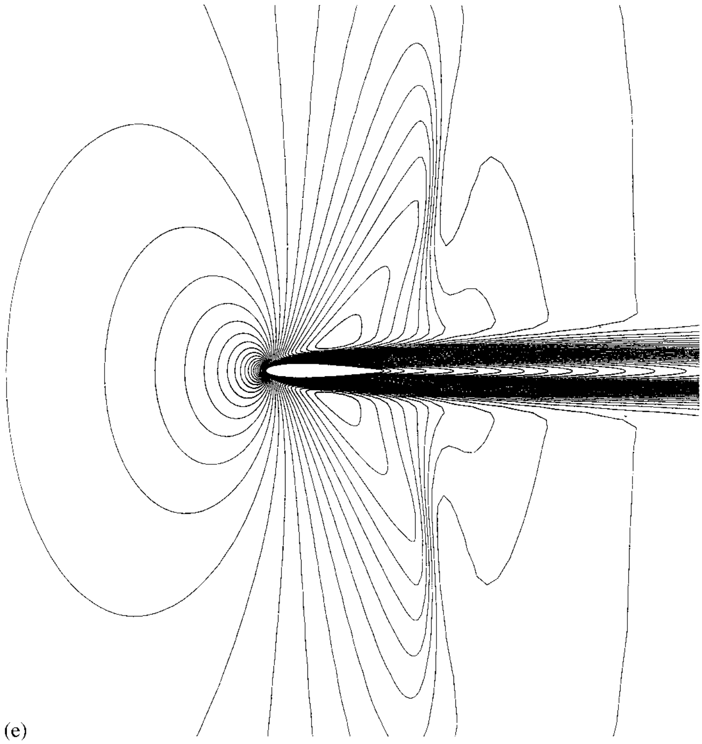

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3. BOUNDARY CONDITIONS

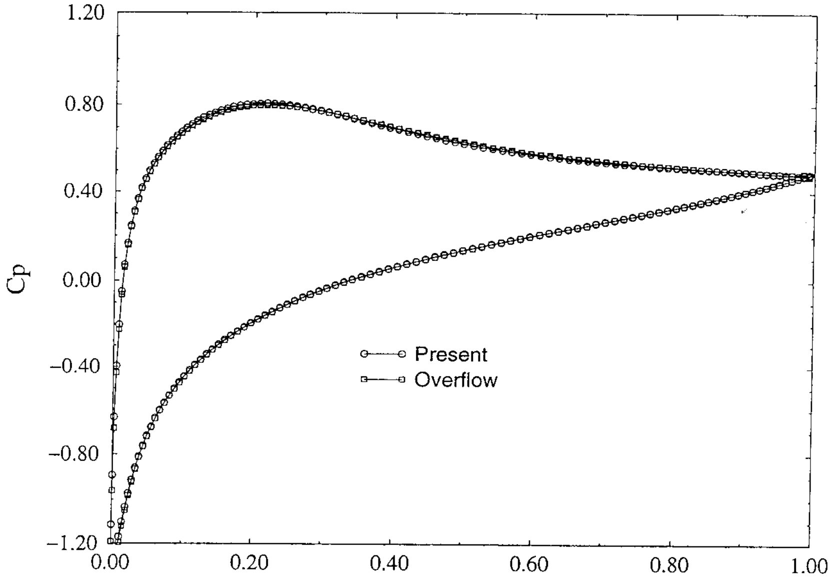
In the far-field the flow is inviscid (except in the wake) and the disturbances decay if the flow is subsonic. The circulation, however, does not vanish, no matter how far the boundary is. There are two conditions at a solid surface, there is no-slip and there is no-penetration. Both conditions are enforced in the solution of the momentum equations in conservation form in the shock region. For smooth flows, the non-conservative form of the momentum equations is used. (Shocks are detected based on the gradient of the density or the pressure.) The above two conditions are imposed in the calculation of the viscous terms in the entropy and vorticity calculations. However, the system of equations for the velocity components admits only the no-penetration boundary condition. The velocity vanishes at the wall only at convergence when the proper value of vorticity is reached.

4. NUMERICAL METHODS

The generalized Cauchy/Riemann equations for the velocity components can be imbedded in an artificial time-dependent symmetric hyperbolic system of the form:

$$W_t = AW_x + BW_y, \tag{21}$$

where $W = (u, v)^t$ and A and B are symmetric matrices.



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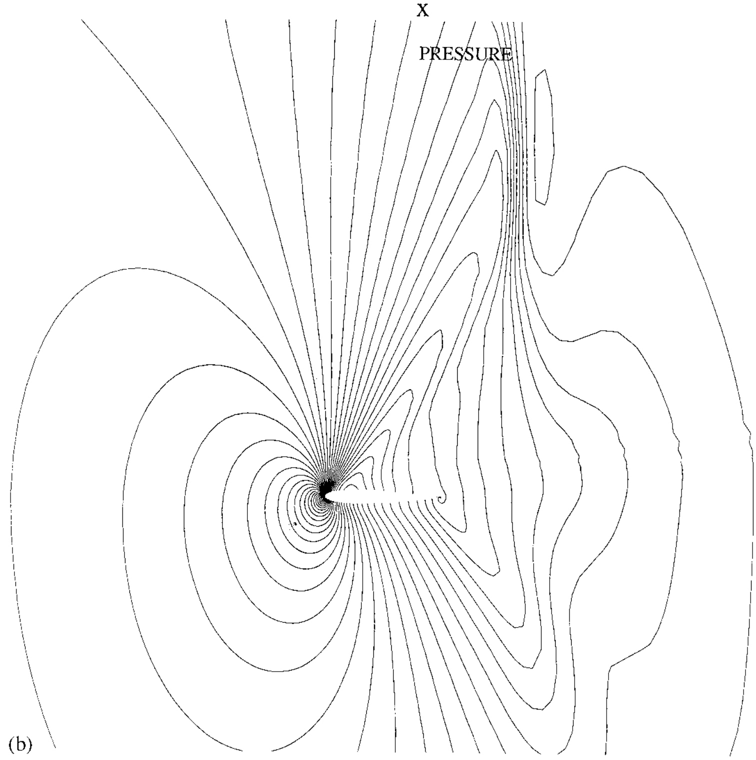

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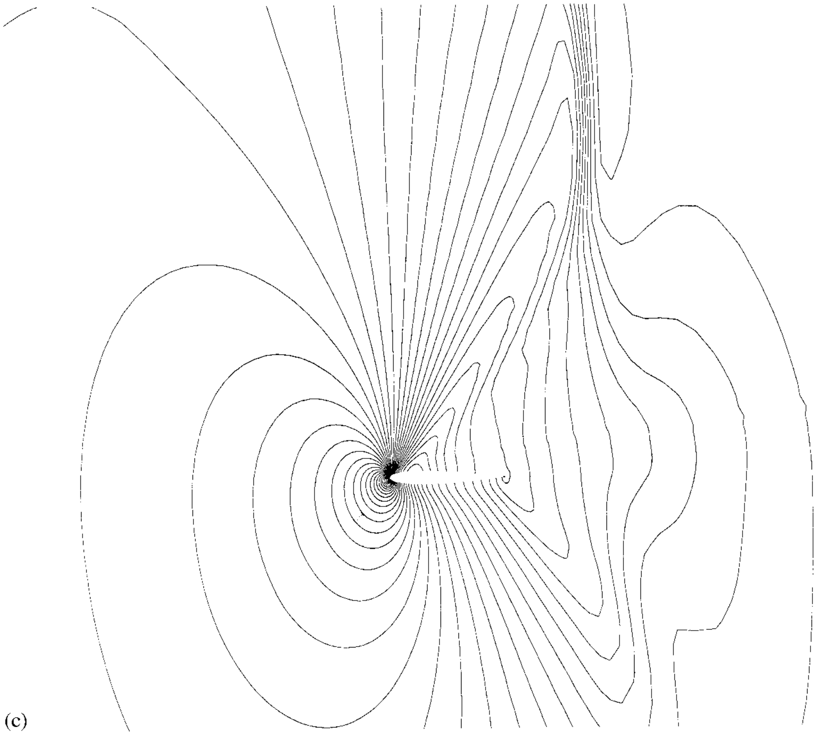
Figure 4. Results for NACA 0012, $\alpha = 10$, $M_{\infty} = 0.9$ and $Re = 500$. (a) Surface pressure distributions; (b) pressure contours (Overflow); (c) pressure contours (present formulation); (d) Mach contours (Overflow); (e) Mach contours (present formulation).



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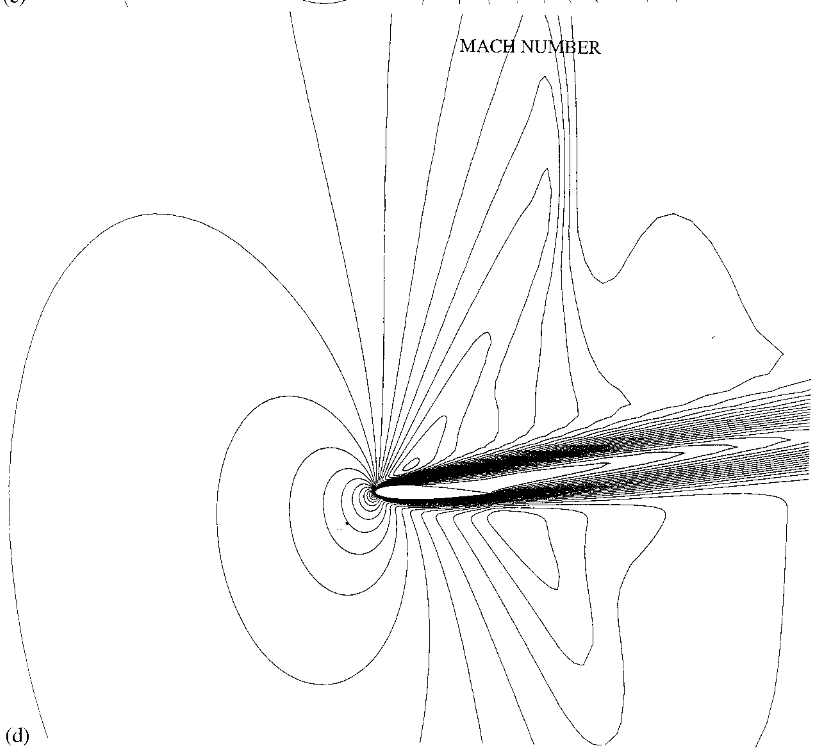

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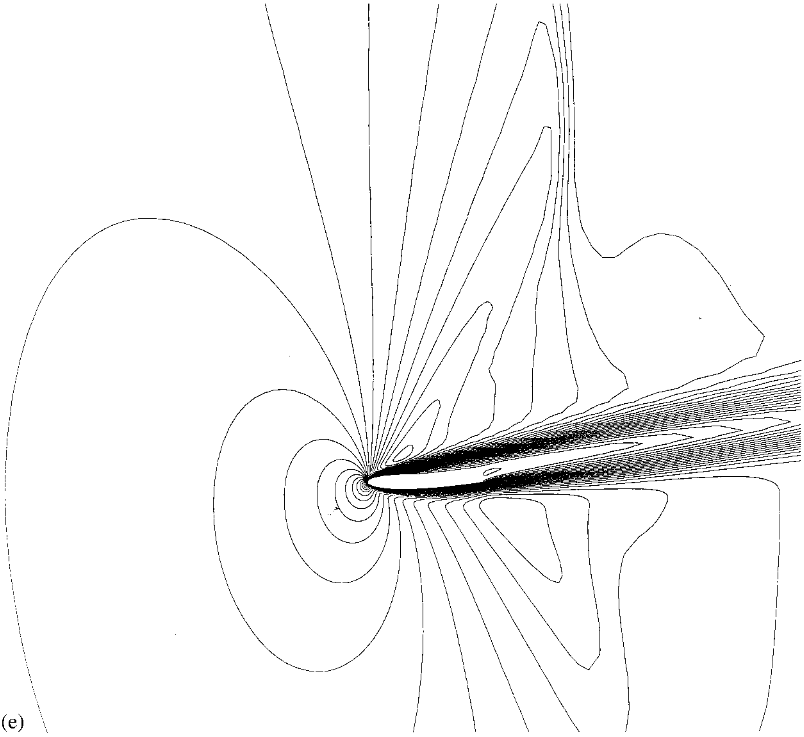


Figure 4 (*Continued*)

One can solve this problem using a combination of second- and fourth-order dissipation, or a least-squares Galerkin method. The following demonstrates how a fourth-order dissipation for smooth flows is constructed. The continuity equation and vorticity definition can be rewritten in the form

$$u_x + v_y = \sigma, \tag{22}$$

$$-u_y + v_x = \omega, \tag{23}$$

where σ can be obtained in terms of the gradient of the density. Differentiating Equation (22) with respect to x and Equation (23) with respect to y, yields

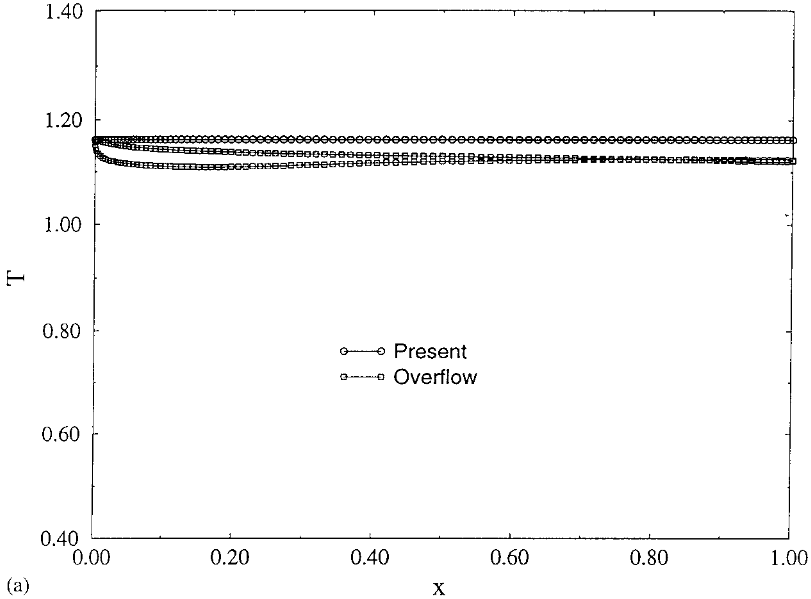
$$\nabla^2 u - \sigma_x + \omega_y = 0 \tag{24}$$

similarly

$$\nabla^2 v - \sigma_y - \omega_x = 0. \tag{25}$$

It turns out that a (second-order) Galerkin approximations of Equations (24) and (25) on structured or unstructured grid (of triangles) are dissipative. Hence the system of Equation (21) is augmented with these terms to avoid decoupling and achieve numerical stability. On the other hand, for the scalar equations of the entropy, standard upwind schemes can be used as well as centered schemes with explicitly added viscosity.

A hyperbolic grid generation code is used to generate a body fitted mesh of C-type. The discrete non-linear algebraic equations are solved via a semi-implicit block relaxation algorithm. No attempt has been made to accelerate the convergence in this feasibility study so



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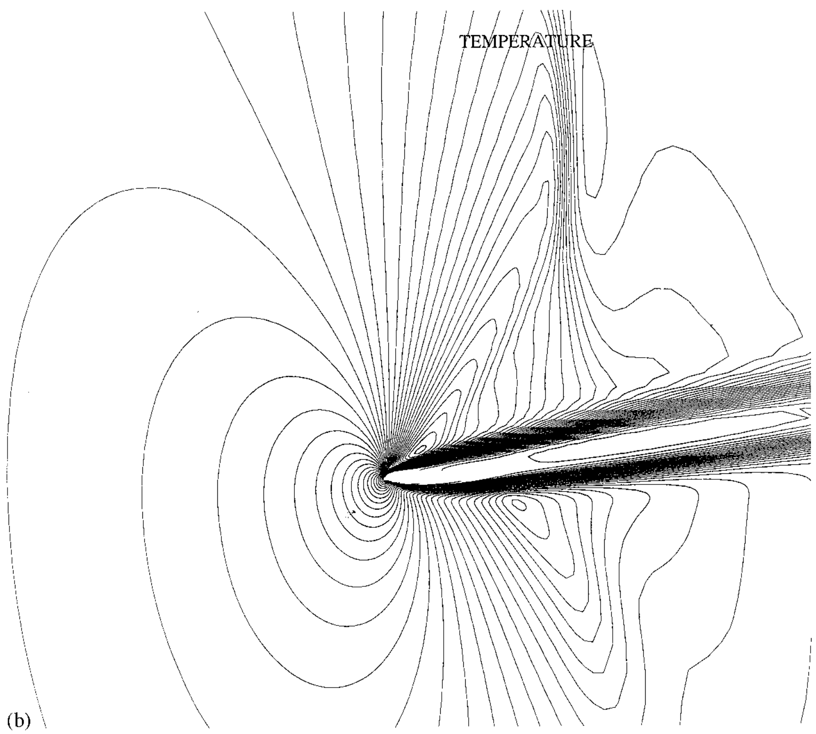
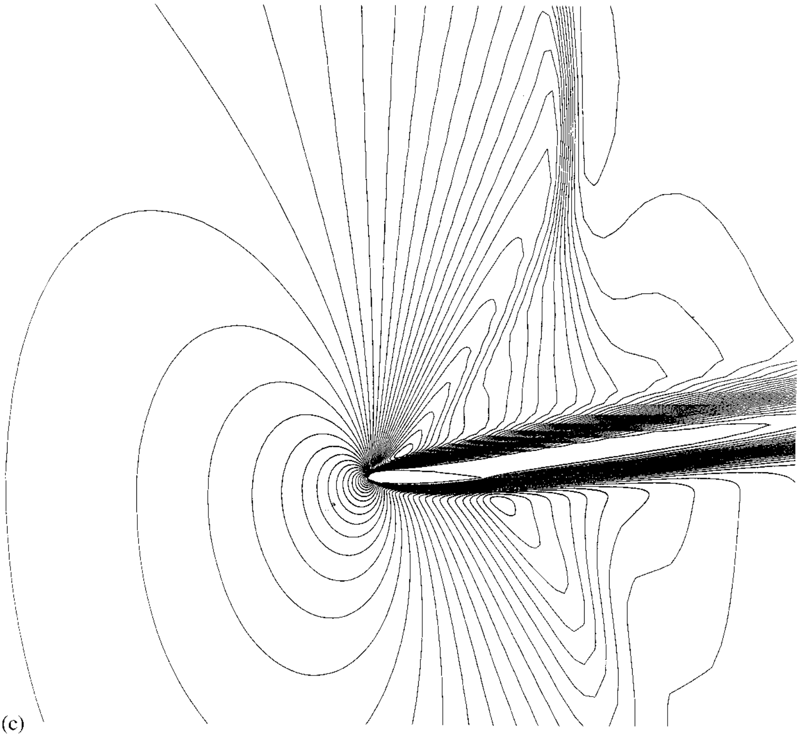


(b)

Figure 5. Results for NACA 0012, $\alpha = 10$, $M_\infty = 0.9$ and $Re = 500$. (a) Surface temperature distributions; (b) temperature contours (Overflow); (c) temperature contours (present formulation).



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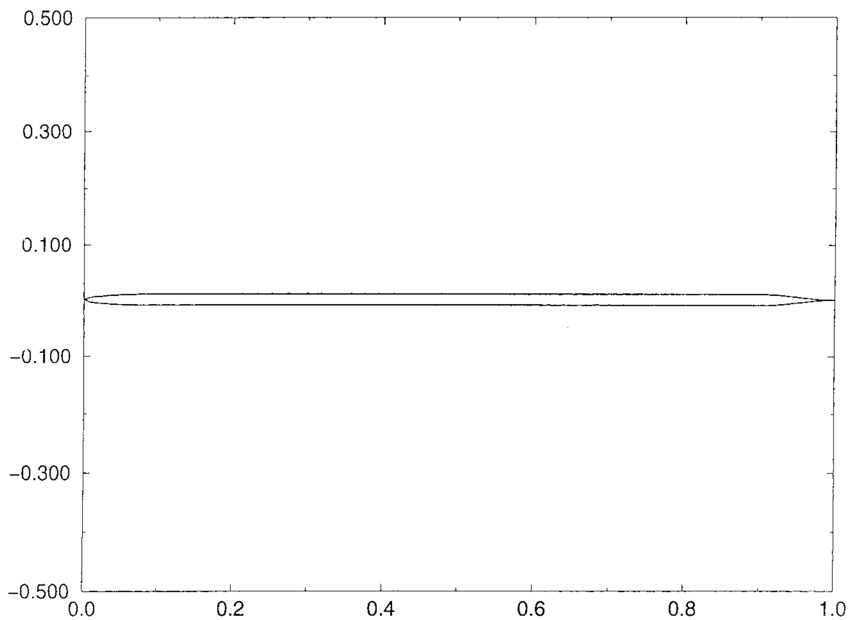

Figure 6. Configuration of thin body.

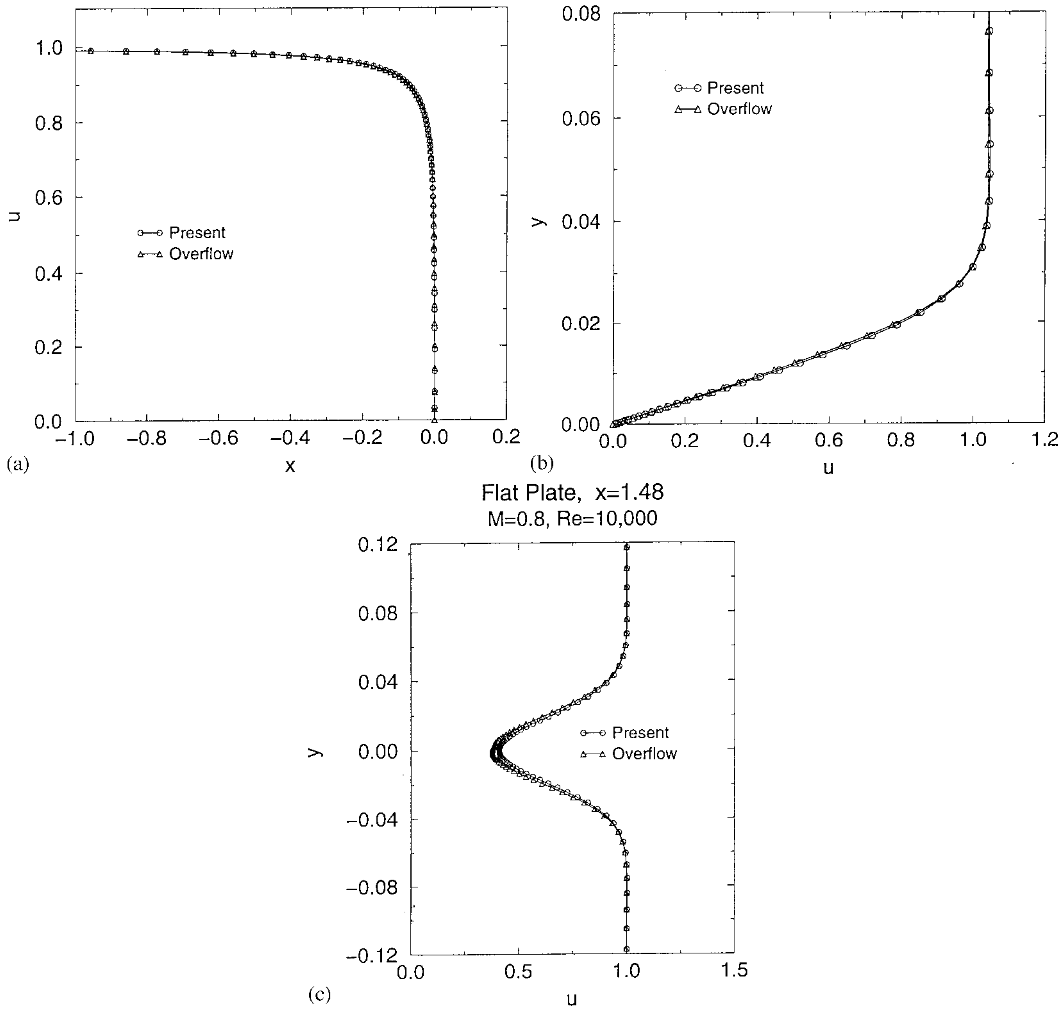

Figure 7. (a) Velocity distribution on the axis. (b) Velocity distributions at the mid-section ($x = 0.5$). (c) Velocity distribution in the wake ($x = 1.48$).

far. The present formulation is amenable for parallel computation via domain decomposition techniques and such implementations will be reported elsewhere.

5. NUMERICAL RESULTS

In this section, the results of the calculations based on the present formulations and the corresponding results using Overflow (NASA Ames code [3]) are presented for compressible laminar flow over NACA 0012 at $M_\infty = 0.8$ and 0.9, at 0° and 10° angles of attack, with $Re = 500$. Figures 1–4 show comparisons of surface pressure distributions and pressure contours and Mach contours for these cases. In Figure 5, the surface temperature distributions and temperature contours are plotted for the last case. The same grid of 301×51 points is used in all these calculations.

The second set of results are for a thin body ($\tau = 2\%$) at $M_\infty = 0.8$ at 0° angle of attack, with $Re = 10000$. Figure 6 shows the configuration. In Figure 7, the velocity distributions are plotted at the axis (in front of the body), at the mid-section ($x = 0.5$) and in the wake ($x = 1.48$).

Unlike Reference [4], no difficulty of convergence is encountered with the present formulation. The maximum residual in any of the above calculations is 10^{-6}. The results for the case of NASA 0012 at 0° angle of attack, $M_\infty = 0\ 8$ and $Re = 500$ agree well with those reported in Reference [5].

This study confirms that the present formulation is equivalent to the standard one. Moreover, assuming the total enthalpy is constant for compressible viscous flow is a reasonable approximation. The temperature on the body surface is slightly different, however, from Overflow calculations, where the body temperature is not necessarily the same as the stagnation temperature.

6. CONCLUDING REMARKS

Steady compressible flows over airfoils are simulated, for low and high Reynolds numbers, based on a non-standard formulation. The velocity components are calculated from a locally preconditioned system consisting of the continuity equation and the vorticity definition, while entropy and the total enthalpy are obtained from two scalar convection, diffusion equations. A further simplification is introduced where the total enthalpy is assumed constant everywhere, even in the viscous layer. Numerical results agree with those obtained, on the same grid, from NASA Ames Overflow code. The present formulation is promising since it is amenable for parallel computations via domain decomposition techniques and adaptive modeling of the flow physics.

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